On Pointed Hopf Algebras with Sporadic Simple Groups HS and Co3

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Abstract

Every non quasi- -1-type Nichols algebra is infinite dimensional. All quasi- -1-type Nichols algebra over sporadic simple groups HS and Co3 are found.

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0 Introduction

This article is to contribute to the classification of finite-dimensional complex pointed Hopf algebras with sporadic simple group $G =$ HS or Co3

Many papers are about the classification of finite dimensional pointed Hopf algebras, for example, [AS02, AS00, AS05, He06, AHS08, AG03, AFZ, AZ07, Gr00, Fa07, AF06, AF07, ZZC, ZCZ, ZZWCY08].

Sporadic simple groups are important in the theories of Lie groups, Lie algebras, vertex operator algebras and algebraic groups. Every non quasi- -1-type Nichols algebra is infinite dimensional (see [AF07, Lemma 1.8, 1.9]). In this paper we find all quasi- -1-type Nichols algebra over sporadic simple groups HS and Co3 by using the results in [ZCH].

For $s \in G$ and $(\rho, V) \in \widehat{G^s}$, here is a precise description of the Yetter-Drinfeld (YD in short) module $M(\mathcal{O}_s, \rho)$, introduced in [Gr00, AZ07]. Let $t_1 = s, \ldots, t_m$ be a numeration of $\mathcal{O}_s$, which is a conjugacy class containing s, and let $g_i \in G$ such that $g_i \triangleright s := g_i s g_i^{-1} = t_i$ for all $1 \leq i \leq m$. Then $M(\mathcal{O}_s, \rho) = \oplus_{1 \leq i \leq m} g_i \otimes V$. Let $g_i v := g_i \otimes v \in M(\mathcal{O}_s, \rho)$, $1 \leq i \leq m$, $v \in V$. If $v \in V$ and $1 \leq i \leq m$, then the action of $h \in G$ and the coaction are given by

$$\delta(g_i v) = t_i \otimes g_i v, \qquad h \cdot (g_i v) = g_j(\gamma \cdot v), \tag{0.1}$$

where $hg_i = g_j\gamma$, for some $1 \le j \le m$ and $\gamma \in G^s$. The explicit formula for the braiding is then given by

$$c(g_iv \otimes g_jw) = t_i \cdot (g_jw) \otimes g_iv = g_{j'}(\gamma \cdot v) \otimes g_iv \tag{0.2}$$

for any $1 \le i, j \le m$, $v, w \in V$, where $t_ig_j = g_{j'}\gamma$ for unique j', $1 \le j' \le m$ and $\gamma \in G^s$. Let $\mathfrak{B}(\mathcal{O}_s, \rho)$ denote $\mathfrak{B}(M(\mathcal{O}_s, \rho))$, which is called a bi-one type Nichols algebra (see [ZZWCY08]). $M(\mathcal{O}_s, \rho)$ is a simple YD module (see [AZ07, Section 1.2]). Furthermore, if χ is the character of ρ, then we also denote $\mathfrak{B}(\mathcal{O}_s, \rho)$ by $\mathfrak{B}(\mathcal{O}_s, \chi)$.

1 Tables about -1- type

In this section all -1- type bi-one Nichols algebras over HS and Co3 up to graded pull-push YD Hopf algebra isomorphisms, are listed in table.

1.1 Quasi-real elements

Definition 1.1. *Let G be a finite group with $s \in G$. s is called a quasi-real element if $s^2 = 1$ or there exists an integer number j such that $s^j \in \mathcal{O}_s$ with $s^j \neq s$. Furthermore, s is called a strongly quasi-real element if $s^2 = 1$ or there exists an integer number j such that s^j, s^{j^2} and s are different each other with $s^j \in \mathcal{O}_s$.*

A Group is said to be quasi-real if its every element is quasi-real. It is clear that every real element is a quasi-real element.

Definition 1.2. *Let G be a finite group with $s \in G$ and s a quasi-real element in G. $\mathfrak{B}(\mathcal{O}_s, \rho)$ with $\rho(s) = q_{ss}\mathrm{id}$ is called to be of quasi- -1 type if one of the following conditions holds.*

(i) *s is a strongly real element and the order of s is even with $q_{ss} = -1$.*

(ii) *s has even order and $\deg(\rho) > 1$ with $q_{ss} = -1$.*

(iii) *$\deg(\rho) = 1$ and the order of s is even with $q_{ss} = -1$*

(iii) $\deg(\rho) = 1$ and q_{ss} is a primitive 3-th root of unity.

By GAP we have the following Lemmas.

Lemma 1.3. *s_i^j, $s_i^{j^2}$ and s_i are different each other with $s_i^j \in \mathcal{O}_{s_i}$ in the following cases:*

(i) *when G is* HS *with* $(i, j) = (8, 3)$, $(9, 3)$, $(10, 2)$, $(12, 3)$, $(13, 3)$, $(14, 3)$, $(15, 2)$, $(17, 2)$, $(22, 2)$, $(23, 3)$, $(24, 3)$.

(ii) *when G is* Co3 *with* $(i, j) = (2, 2)$, $(3, 2)$, $(4, 2)$, $(5, 2)$, $(7, 5)$, $(8, 2)$, $(14, 3)$, $(15, 3)$, $(16, 3)$, $(17, 2)$, $(19, 2)$, $(20, 2)$, $(22, 3)$, $(23, 3)$, $(33, 3)$, $(34, 3)$, $(35, 3)$, $(36, 3)$, $(38, 2)$, $(39, 7)$, $(42, 2)$.

Lemma 1.4. (i) *If G is* HS*, then s_i is a strongly quasi-real element except* $i=2,\ 3,\ 5,\ 6,\ 11, 16,\ 18,\ 19,\ 20,\ 21$.

(ii) *If G is* Co3*, then s_i is a strongly quasi-real element except* $i=6,\ 9, 10,\ 12,\ 18,\ 21, 24,\ 25, 26, 27,\ 28, 29, 30,\ 31, 32,\ 37, 40, 41$.

Lemma 1.5. *s_i^j and s_i are different each other with $s_i^j \in \mathcal{O}_{s_i}$ in the following cases:*

(i) when G is HS *with* $(i,j) = (2,3),\ (3,3),\ (5,3),\ (6,3),\ (8,3),\ (9,2),\ (10,2),\ (11,2),\ (12,3),\ (13,3),\ (14,3),\ (15,2),\ (16,3),\ (17,2),\ (18,5),\ (19,5),\ (20,5),\ (21,3),\ (22,2),\ (23,3),\ (24,3)$.

(ii) *when G is* Co3 *with* $(i,j) = (2,2),\ (3,2),\ (4,2),\ (5,2),\ (6,2),\ (7,5),\ (8,2),\ (9,2),\ (10,5),\ (12,2),\ (14,3),\ (15,3),\ (16,3),\ (17,2),\ (18,3),\ (19,2),\ (20,2),\ (21,2),\ (22,3),\ (23,3),\ (24,3),\ (25,3),\ (26,3),\ (27,5),\ (28,5),\ (29,5),\ (30,5),\ (31,3),\ (32,5),\ (33,3),\ (34,3),\ (35,3),\ (36,3),\ (37,5),\ (38,2),\ (39,7),\ (40,5),\ (41,5),\ (42,2)$.

Lemma 1.6. HS *and* Co3 *are quasi-real.*

1.2 Quasi--1 types

In Table 1–3, we use the following notations. s_i denotes the representative of i-th conjugacy class of G (G is HS and Co3); $\chi_i^{(j)}$ denotes the j-th character of G^{s_i} for any i; $\nu_i^{(1)}$ denotes the number of conjugacy classes of the centralizer G^{s_i}; $\nu_i^{(2)}$ denote the number of character $\chi_i^{(j)}$ of G^{s_i} with non -1-type $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$; $\mathrm{cl}_i[j]$ denote that s_i is in j-th conjugacy class of G^{s_i}.

Using the results in [ZCH] we can obtain the following tables.

HS					
s_i	$\mathrm{cl}_i[p]$	Order(s_i)	j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of quasi- -1 type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_1	$\mathrm{cl}_1[1]$	1		24	24
s_2	$\mathrm{cl}_2[8]$	8	3,4	16	14
s_3	$\mathrm{cl}_3[21]$	4	9,10,11,12,13,14,15,16,17,18	22	12
s_4	$\mathrm{cl}_4[2]$	2	12, 24, 25, 27, 28	28	23
s_5	$\mathrm{cl}_5[11]$	8	2,4	16	14
s_6	$\mathrm{cl}_6[30]$	4	5,6,7,8,13,14,15,16,17,20,24,25,26	34	21
s_7	$\mathrm{cl}_7[2]$	2	2, 3, 4, 5. 9, 10, 11, 12, 19, 20, 23, 24, 26	26	13
s_8	$\mathrm{cl}_8[3]$	10	2,4	20	18
s_9	$\mathrm{cl}_9[23]$	5		25	25
s_{10}	$\mathrm{cl}_{10}[7]$	15		15	15
s_{11}	$\mathrm{cl}_{11}[21]$	3	3,4,5,6	21	17
s_{12}	$\mathrm{cl}_{12}[4]$	20	2	20	19
s_{13}	$\mathrm{cl}_{13}[2]$	20	2	20	19
s_{14}	$\mathrm{cl}_{14}[5]$	10	3,4	20	18
s_{15}	$\mathrm{cl}_{15}[25]$	5		26	26
s_{16}	$\mathrm{cl}_{16}[4]$	4	8,9,10,11,16,17,24	26	19
s_{17}	$\mathrm{cl}_{17}[4]$	7		7	7
s_{18}	$\mathrm{cl}_{18}[10]$	12	2,3,5	12	9
s_{19}	$\mathrm{cl}_{19}[10]$	6	5,6,7,8,9,10,11,12,13	15	6
s_{20}	$\mathrm{cl}_{20}[16]$	6	2,4,7,8,11,12,13	18	11
s_{21}	$\mathrm{cl}_{21}[2]$	8	2,3	16	14
s_{22}	$\mathrm{cl}_{22}[17]$	5		25	25
s_{23}	$\mathrm{cl}_{23}[4]$	11		11	11
s_{24}	$\mathrm{cl}_{24}[8]$	11		11	11

Table 1

CO_3					
s_i	$\mathrm{cl}_i[p]$	Order(s_i)	j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of quasi- -1 type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_1	$\mathrm{cl}_1[1]$	1		42	42
s_2	$\mathrm{cl}_2[4]$	23		23	23
s_3	$\mathrm{cl}_3[14]$	23		23	23
s_4	$\mathrm{cl}_4[2]$	15		15	15
s_5	$\mathrm{cl}_5[25]$	5		25	25
s_6	$\mathrm{cl}_6[3]$	3		48	48
s_7	$\mathrm{cl}_7[7]$	18	2	18	17
s_8	$\mathrm{cl}_8[23]$	9		30	30
s_9	$\mathrm{cl}_9[3]$	3		60	60
s_{10}	$\mathrm{cl}_{10}[4]$	6	2,3,4,8,11,12,14,40,43,44	48	38
s_{11}	$\mathrm{cl}_{11}[2]$	2	3, 10, 12, 13, 19, 20, 22, 24, 33, 39, 41, 42, 43	43	30
s_{12}	$\mathrm{cl}_{12}[44]$	6	5,6,7,8,9,10,11,12,15,26,27,36,37,41	45	31
s_{13}	$\mathrm{cl}_{13}[2]$	2	2, 3, 4, 7, 8, 11, 13, 16, 17, 18, 22, 24, 26, 28, 30	30	15
s_{14}	$\mathrm{cl}_{14}[10]$	20	2	20	19
s_{15}	$\mathrm{cl}_{15}[14]$	20	2	20	19
s_{16}	$\mathrm{cl}_{16}[12]$	10	3,4,21	30	27
s_{17}	$\mathrm{cl}_{17}[2]$	5		32	32
s_{18}	$\mathrm{cl}_{18}[4]$	4	6,11,12,13,14,17,18,23,25,32,33	37	26
s_{19}	$\mathrm{cl}_{19}[5]$	21		21	21
s_{20}	$\mathrm{cl}_{20}[19]$	7		21	21
s_{21}	$\mathrm{cl}_{21}[3]$	3	4,5,6,7,8,9	33	27
s_{22}	$\mathrm{cl}_{22}[4]$	14	2	14	13
s_{23}	$\mathrm{cl}_{23}[19]$	10	2,4	20	18
s_{24}	$\mathrm{cl}_{24}[2]$	8	2,3,4,5	20	16
s_{25}	$\mathrm{cl}_{25}[3]$	4	5,6,7,8,11,12,17,18,19,20,25,26, 27,28,35,36,39,40,43,44	50	30
s_{26}	$\mathrm{cl}_{26}[29]$	8	3,4,9,21,24	32	27

Table 2

CO_3					
s_i	$\mathrm{cl}_i[p]$	Order(s_i)	j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of quasi- -1 type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_{27}	$\mathrm{cl}_{27}[16]$	24	2,3,5	24	21
s_{28}	$\mathrm{cl}_{28}[21]$	12	5,6,7,8,17,18,19,20,21,22,23,24	30	18
s_{29}	$\mathrm{cl}_{29}[6]$	6	3,4,5,6,7,8,9,10,44,49	51	41
s_{30}	$\mathrm{cl}_{30}[20]$	24	2,3,5	24	21
s_{31}	$\mathrm{cl}_{31}[32]$	8	2,4,10,22,23	32	27
s_{32}	$\mathrm{cl}_{32}[17]$	6	2,4,6,8,11,12	20	14
s_{33}	$\mathrm{cl}_{33}[13]$	22	2	22	21
s_{34}	$\mathrm{cl}_{34}[8]$	22	2	22	21
s_{35}	$\mathrm{cl}_{35}[21]$	11		22	22
s_{36}	$\mathrm{cl}_{36}[12]$	11		22	22
s_{37}	$\mathrm{cl}_{37}[17]$	12	2,5,6,9,10,13,14,15,16,37	42	32
s_{38}	$\mathrm{cl}_{38}[19]$	15		30	30
s_{39}	$\mathrm{cl}_{39}[29]$	30	2	30	29
s_{40}	$\mathrm{cl}_{40}[24]$	12	2,5,6,7,8,15,16,17,18	36	27
s_{41}	$\mathrm{cl}_{41}[23]$	6	2,5,6,9,10,13,14,15,16,19	24	14
s_{42}	$\mathrm{cl}_{42}[33]$	9		33	33

Table 3

Table 1 is called the table of HS and Table 2–3 are called the tables of Co3.

2 Bi-one Nichols algebras over HS and Co3

In this section all -1-type bi-one Nichols algebra over HS and Co3 of exceptional type up to graded pull-push YD Hopf algebra isomorphisms are given.

Lemma 2.1. *Assume that $s \in G$ is quasi-real with $\rho \in \widehat{G^s}$. If $\mathfrak{B}(\mathcal{O}_s, \rho)$ is not of quasi--1 type, then* $\dim \mathfrak{B}(\mathcal{O}_s, \rho) = \infty$.

Proof. If follows from [AF07, Lemma 1.8, 1.9]. □.

We give our main result.

Theorem 1. *Let G be one of* HS *and* Co3.

(i) $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ *is of -1-type if and only if j appears in the fourth column of the table of G.*

(ii) $\dim(\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})) = \infty$ *if j does not appears in the fourth column of the table of G.*

Proof. (i) It follows from the program.

(ii) It follows from Lemma 2.1. □

3 Appendix

In this section Suzuki group Sz(8) is considered.

Lemma 3.1. (i) s^j, s^{j^2} *and* s *are different each other with* $s^j \in \mathcal{O}_s$ *in the following cases:*

$(i, j) = (5, 5)$, $(6, 5)$, $(7, 5)$, $(11, 2)$.

(ii) s_i *is strongly quasi-real except* $i = 2, 3, 4, 8, 9$.

(iii) s_i *is strongly quasi-real if and only if* s_i *is quasi-real.*

Sz(8)					
s_i	$\mathrm{cl}_i[p]$	$Order(s_i)$	j such that $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ is of quasi- -1 type	$\nu_i^{(1)}$	$\nu_i^{(2)}$
s_1	$\mathrm{cl}_1[1]$	1		11	11
s_2	$\mathrm{cl}_2[8]$	7		7	7
s_3	$\mathrm{cl}_3[21]$	7		7	7
s_4	$\mathrm{cl}_4[2]$	7		7	7
s_5	$\mathrm{cl}_5[11]$	13		7	7
s_6	$\mathrm{cl}_6[30]$	13		13	13
s_7	$\mathrm{cl}_7[2]$	13		13	13
s_8	$\mathrm{cl}_8[3]$	4		16	16
s_9	$\mathrm{cl}_9[23]$	4		16	16
s_{10}	$\mathrm{cl}_{10}[7]$	2	9, 10, 13, 14, 17, 18, 19, 20	22	14
s_{11}	$\mathrm{cl}_{11}[21]$	5		5	5

Table 4

Proposition 3.2. *Let* G *be* Sz(8) *with* $i = 1, 5, 6, 7, 10, 11$.

(i) $\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})$ *is of quasi-*-1*-type if and only if* j *appears in the fourth column of the table of Table 4.*

(ii) $\dim(\mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)})) = \infty$ *if* j *does not appears in the fourth column of the table of Table 4.*

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